PROJECT

Numerical models for geothermal energy exploitation in volcanic regions

Instructor
Armando Coco

Students
Mattu Riccardo  Marini Fabio

Catania
2024
Summary

- Physical problem
- Mathematical model
- Geothermal values
- Discretization
- Numerical results
- Conclusions and future works
- References
Physical problem

Figure: Geothermal power station
Mathematical model: equations

Equations

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \alpha \Delta T = 0 \quad (\text{convection-diffusion equation}) \]

\[ \mathbf{u} = -\frac{k}{\mu} \nabla p \quad \text{Darcy Law} \]

\[ \text{div}(\mathbf{u}) = 0 \quad \text{Conservation of mass} \]
Mathematical model: Boundary condition

- Dirichlet BCs
- Neumann BCs
- Temperature
- Magmatic chamber
Mathematical model: Temperature at $t = 0$

Figure: Plot of the temperature at the initial time
### Geothermal values

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>( \alpha )</th>
<th>( \nu )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical meaning</td>
<td>diffusivity</td>
<td>permeability</td>
<td>viscosity</td>
</tr>
<tr>
<td>Units</td>
<td>( m^2 \cdot s^{-1} )</td>
<td>( m^2 )</td>
<td>( Pa \cdot s^{-1} )</td>
</tr>
<tr>
<td>I scenario</td>
<td>( 1.43 \cdot 10^{-7} )</td>
<td>( 10^{-14} )</td>
<td>( 8.9 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>II scenario</td>
<td>( 2.3 \cdot 10^{-5} )</td>
<td>( 10^{-14} )</td>
<td>( 2.2 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>III scenario</td>
<td>( 0.5 \cdot 10^{-5} )</td>
<td>( 10^{-14} )</td>
<td>( 0.6 \cdot 10^{-3} )</td>
</tr>
</tbody>
</table>

- **Diffusivity:** This constant indicates how significant the diffusion is. The greater the alpha, the more the heat spreads.

- **Permeability:** is the ability of underground rocks to allow fluid to pass through.

- **Viscosity:** in a fluid are defined as those resulting from the relative velocity of different fluid particles.
Discretization: Explicit vs Implicit

We implemented two types of discretization for time dependance:

**Equations (Forward Euler)**

\[
\frac{T^{n+1} - T^n}{\tau} + u \nabla T^n - \alpha \Delta T^n = 0
\]

**Equations (Backward Euler)**

\[
\frac{T^{n+1} - T^n}{\tau} + u \nabla T^{n+1} - \alpha \Delta T^{n+1} = 0
\]

Where \( T^n \) is \( T \) evaluate at time \( t_0 + n \cdot dt \).

The explicit method is easy to implement but has significant restrictions on temporal stepsize.

On the other hand, the implicit one is more robust but requires solving a linear system for each iteration.
Discretization: Advection term

1. *Central Finite Difference*, is a second-order method, but is affected by oscillation.

2. *Upwind scheme*, is a first-order method but more stable.

### Equations

\[
u \cdot \nabla T = u_1 (\partial_x T) + u_2 (\partial_y T)
\]

1. \[\partial_x T = \frac{T_{i+1,j} - T_{i-1,j}}{2 \, dx} + o(||dx||^2)\]

2. \[\partial_x T = \begin{cases} \frac{T_{i+1,j} - T_{i,j}}{dx} & \text{if } u_1 < 0 \\ \frac{T_{i-1,j} - T_{i,j}}{dx} & \text{if } u_1 > 0 \end{cases} + o(||dx||)\]

Where \(T_{ij} = T(x_i, y_j, \bar{t})\) and \(\bar{t}\) is fixed. We can use *central differences* when the Péclet condition is satisfied: \(u_1 \, dx < 2\alpha\).
We used a **second-order** scheme in space. For the Laplace operator we used classical *Central finite difference*:

\[
\alpha \Delta T = \alpha \left( \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{dx^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{dy^2} \right) \\
+ o\left(\| (dx, dy) \|^2 \right)
\]

for Neumann boundary condition we used *one-sided finite difference*

\[
-3T_{ij} + 4T_{ij+1} - T_{ij+2} \\ 2dy = g\hat{n}
\]

where $g\hat{n}$ is the normal component on the boundary.
Discretization: Darcy law

Since $\nu$ and $\mu$ are constant, the Darcy law and the conservation of mass read

$$\text{div}\left(-\frac{\nu}{\mu} \nabla p\right) = -\frac{\nu}{\mu} \text{div}(\nabla p) = 0 \Rightarrow \Delta p = 0$$

the discretization is similar to the Laplacian of Temperature. Moreover, we can approximate the velocity vector field $u$ using the \textit{central finite difference}

$$u_1 = -\frac{\nu}{\mu} \left(\frac{p_{i+1j} - p_{i-1j}}{2dx}\right) \quad u_2 = -\frac{\nu}{\mu} \left(\frac{p_{ij+1} - p_{ij-1}}{2dy}\right)$$

The velocity does not depend on time; it is a stationary flow. We solve these equations only once and then solve the heat equation.
Numerical results: III scenario after 90 days
Numerical results: Evolution

Figure: From left to right: Temperature after 30, 60, 90 days
Numerical results: Animated Evolution

Temperature

[Temperature graph with color scale and axes]

ECMI EUROPEAN CONSORTIUM FOR MATHEMATICS IN INDUSTRY

Matt Riccardo Marini Fabio

June 28, 2024
Numerical result: Central F.D. vs Upwind
Conclusions and Future works

Conclusions

▶ $dx \neq dy$
▶ Upwind vs Central Finite Difference
▶ spreading of heat fluid
▶ range of depth

Future works

▶ $\mu = \mu(T(x))$
▶ $\nu = \nu(x)$
▶ II order time (ex. Cranck Nicolson)

A. Allendes, G. Campana, F. Fuicas, E. Otarola: ”Darcy’s problem coupled with the heat equation under singular forcing; analysis and discretization”.


Thanks for your attention!