Many-agent systems in swarm additive manufacturing: How can we uniformly cover a portion of a domain?

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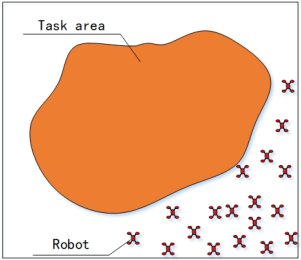
Introduction to the problem

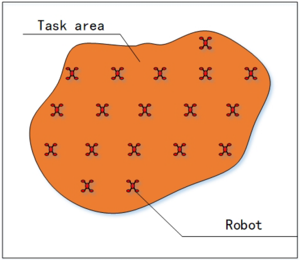
Informal description: Given a set of agents (robots, particles, etc.) and a target domain, we want to steer the robots to cover the domain uniformly.

Applications: For example:

- **Additive manufacturing (3D-printing)**
- Environmental monitoring
- Search and rescue operations
- Military surveillance
- Marine exploration



(a) Before swarm robot coverage



(b) After swarm robot coverage

Given some domain

$$\Omega \in \mathbb{R}^d,$$

and a set of N agents identified by their spatial positions

$$\{X_1, \dots, X_N\},$$

we model their movement using the system of stochastic differential equations (SDEs)

$$dX_i = \mu(X_i, t)\, dt + \sigma(X_i, t)\, dW_i^t, \quad i = 1, \dots, N.$$

The goal is to design a drift term μ and a diffusion term σ so that the agents cover Ω uniformly.

Let $d = 1$ and

$$\Omega = [c_0 - \delta, c_0 + \delta].$$

An intuitive strategy is to let the agents drift towards the domain when outside of the domain, with no drift while inside.

One way to achieve this is by the drift term

$$\mu(X_i, t) = (c_0 - X_i) \cdot \mathbb{1}(X_i \notin \Omega), \quad i = 1, \dots, N,$$

where $\mathbb{1}(\cdot)$ denotes the indicator function.

Another way to achieve this is by the drift term

$$\mu(X_i, t) = \left[\operatorname*{arg\,min}_{b \in \partial\Omega} ||b - X_i||_2 - X_i\right] \cdot \mathbb{1}(X_i \notin \Omega), \quad i = 1, \ldots, N.$$

In the first case, we have attraction to the center, while in the second case the attraction is to the border.

We define the empirical density as

$$f^N(x,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i)$$

As $N \to \infty$, we can treat the particles as densities satisfying the Fokker-Planck equation

$$\frac{\partial f^\infty(x,t)}{\partial t} = \nabla \cdot \left[\mu(x,t)\cdot f^\infty(x,t) + \sigma(x,t)\nabla_x f^\infty(x,t)\right].$$

Steady States

For the simple case of a circular domain, we have analytical solutions for the problem $\frac{\partial f^\infty}{\partial t} = 0$. For the case of attraction to the center, we have,

$$h(x) = \begin{cases} \dfrac{m_2}{2\delta}, & x \in \Omega \\ m_1 \exp\left(-\dfrac{(x-x_0)^2}{2\sigma^2}\right), & x \notin \Omega, \end{cases}$$

and for the case of attraction to the border, we have

$$h(x) = \begin{cases} \dfrac{m_2'}{2\delta}, & x \in \Omega \\ m_1' \exp\left(-\dfrac{(x-x_0)^2 + 2\delta|x-x_0|}{2\sigma^2}\right), & x \notin \Omega, \end{cases}$$

for some constants m_1, m_2, m_1' and m_2'.

Steady States

The steady states must follow continuity and mass conditions:

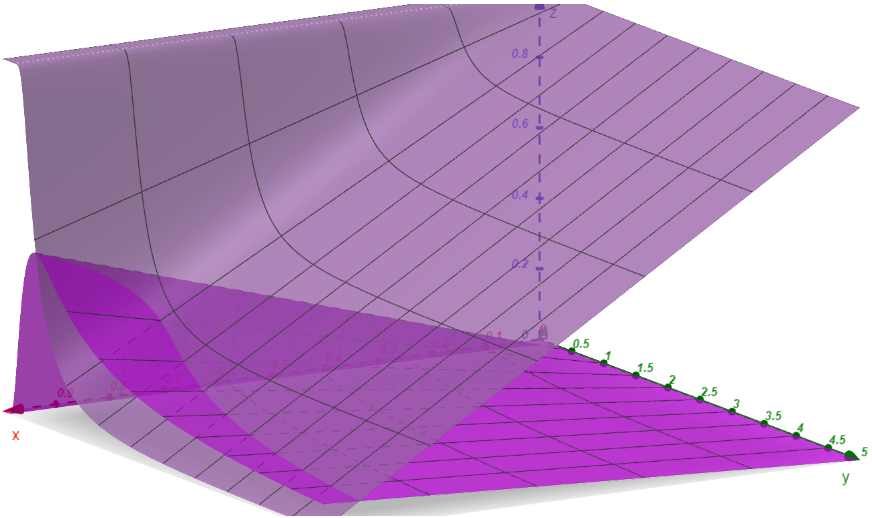
$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} h(x)dx = 1 \\ \displaystyle\lim_{x\to\partial\Omega^+} h(x) = h|_\Omega \end{cases}$$

The systems become:

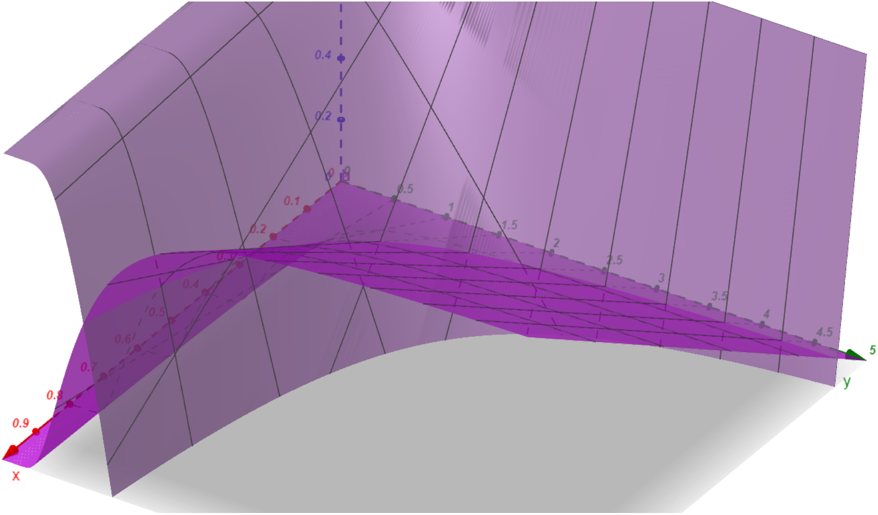
$$\begin{cases} m_2 + m_1 \left[1 + \operatorname{erf}\left(-\dfrac{\delta^2}{2\sigma^2}\right)\right] - 1 = 0 \\ \dfrac{m_1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{\delta^2}{2\sigma^2}\right) - \dfrac{m_2}{2\delta} = 0 \end{cases}$$

$$\begin{cases} m_2' + m_1'\sigma\sqrt{2\pi} \exp\left(\dfrac{\delta^2}{2\sigma^2}\right) \left[1 + \operatorname{erf}\left(-\dfrac{2\delta}{\sqrt{2}\sigma}\right)\right] - 1 = 0 \\ m_1' \exp\left(-\dfrac{3\delta^2}{2\sigma^2}\right) - \dfrac{m_2'}{2\delta} = 0 \end{cases}$$

Finding Stationary Coefficients



Finding Stationary Coefficients



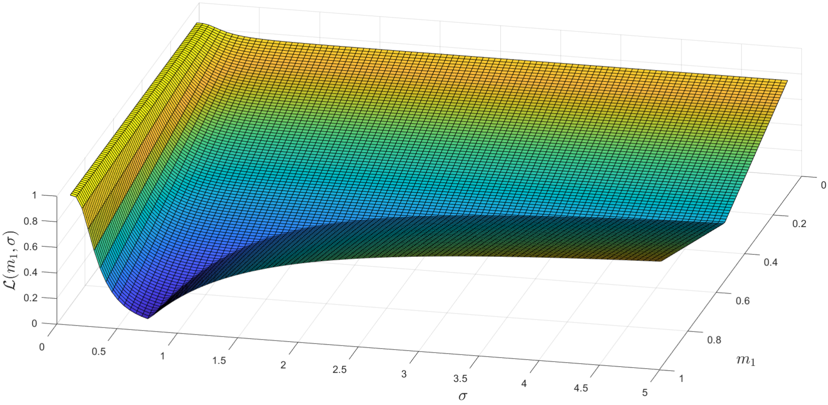
The problem that arises is that o finding a σ that satisfies a given mass (m_2) inside Ω.
Fixing m_2, we get a non linear system in m_1 and σ. We use Newton's method with initial estimates given by Simulated Annealing, to find the estimates.
The risk functional for the Simulated Annealing is:

$$\mathcal{L}(m_1, \sigma) = |F_1(m_1, \sigma)| + |F_2(m_1, \sigma)|$$

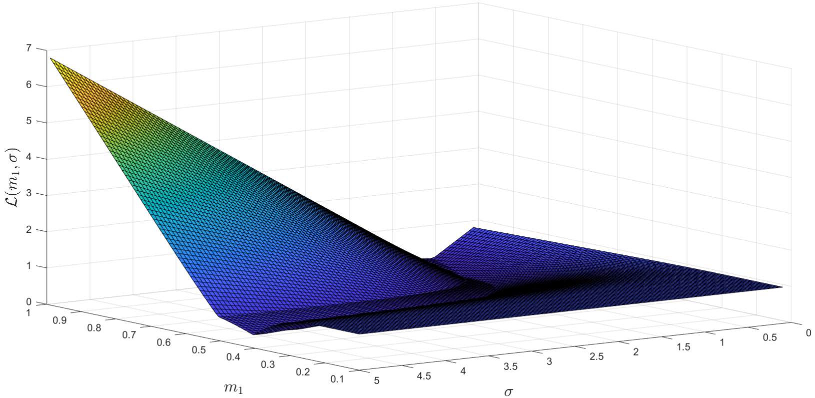
With:

$$\begin{cases} F_1 := \int_{\mathbb{R}^d} h(x)dx - 1 \\ F_2 := \lim_{x \to \partial\Omega^+} h(x) - h|_\Omega \end{cases}$$

Finding Stationary Coefficients

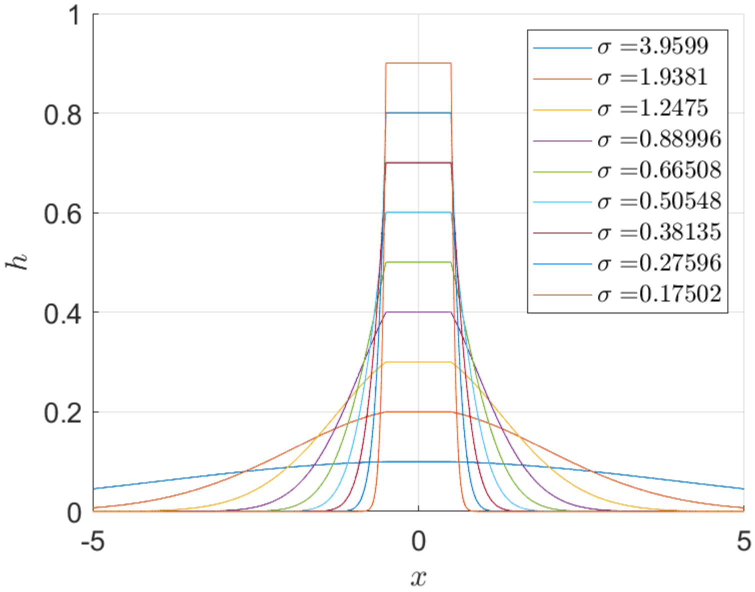


Finding Stationary Coefficients

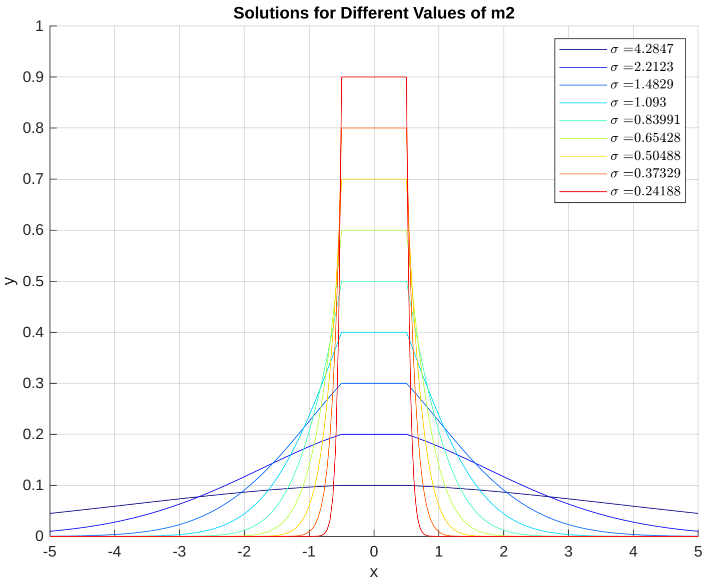


Border atracted loss functional

Finding Stationary Coefficients



Finding Stationary Coefficients



Using the Euler-Maruyama method, which is based on the recursive scheme

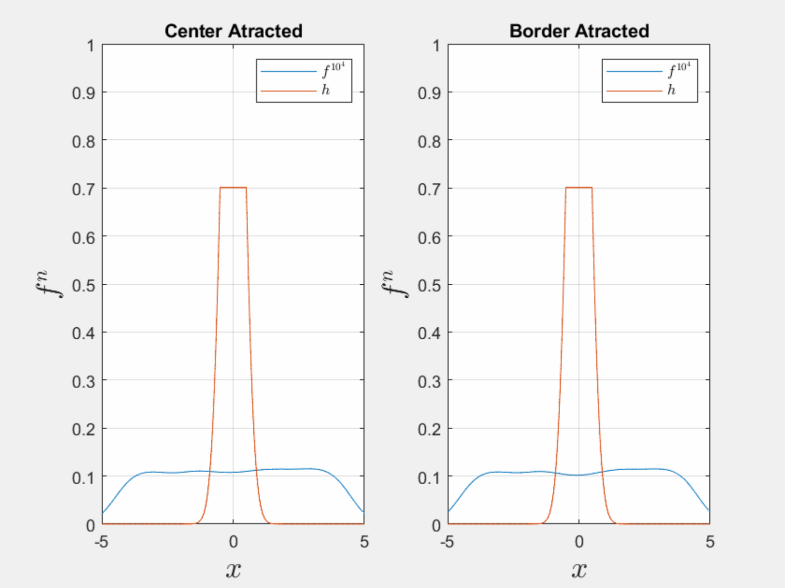
$$X^{k+1} = X^k + \mu(X^k, t_k)\,\Delta t + \sigma(X^k, t_k)\,\Delta W^k,$$

where

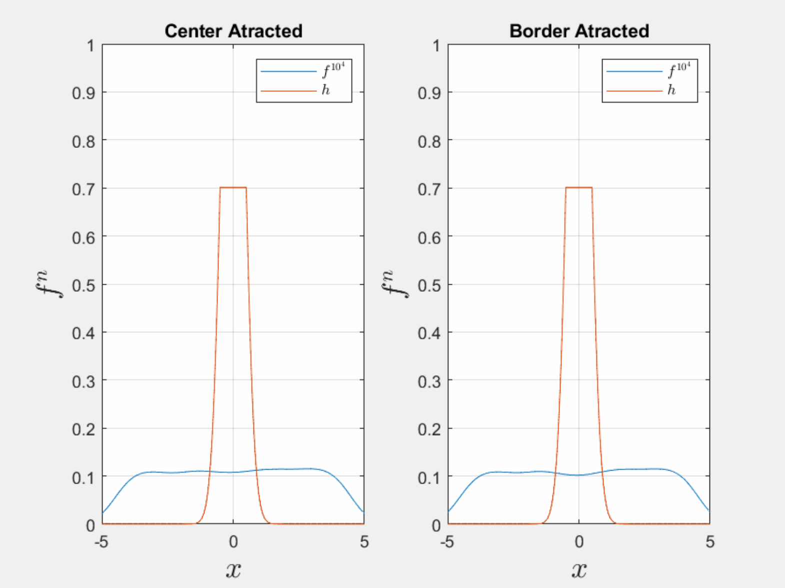
$$\Delta W^k = W_{t_{k+1}} - W_{t_k} \sim \mathcal{N}(0, t_{k+1} - t_k)$$

we can get a numerical estimate for the steady-state distribution.

Numerical experiments (one-dimensional)

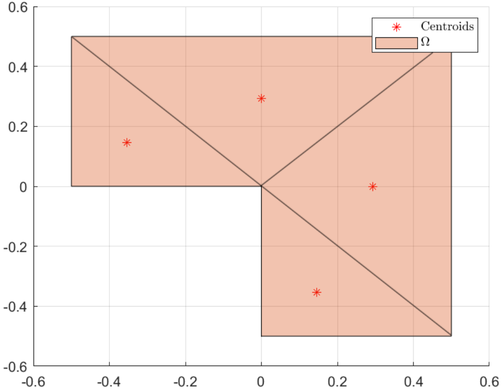


Numerical experiments (one-dimensional)



Two-dimensional non-convex domains

As a non convex domain doesn't have a center in the conventional sense, we divide these domains with Delaunay triangulation and use the triangulation centroids as the centers of atraction.



With $\mathcal{C} = \{c_1, ..., c_m\}$ being the set of centroids, the dynamics for this problem become:

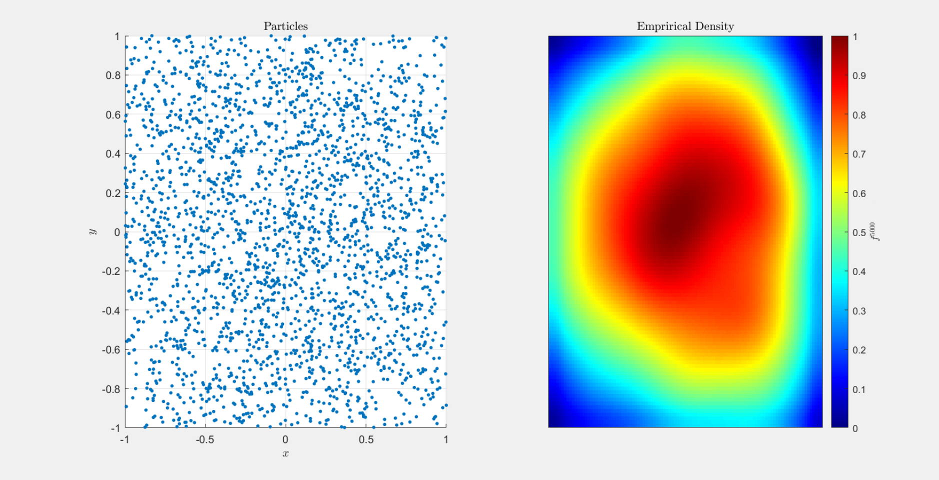
$$dX_i = \left[\underset{c \in \mathcal{C}}{\arg\min} \|c - X_i\|_2 - X_i\right] \mathbb{1}(x \notin \Omega)dt + g(t)dW_t^i$$

The function $g(t)$ is a decaying function of the form:

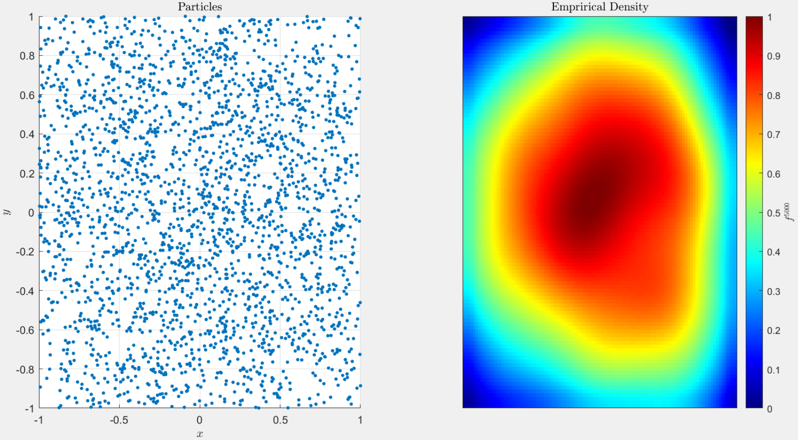
$$g(t) = \sigma_0 \exp\left(-\lambda t\right) + \sigma_T$$

This allows for the system to start with σ_0 and tend towards a more stable Brownian motion σ_T.

Numerical experiments (two-dimensional)



Numerical experiments (two-dimensional)



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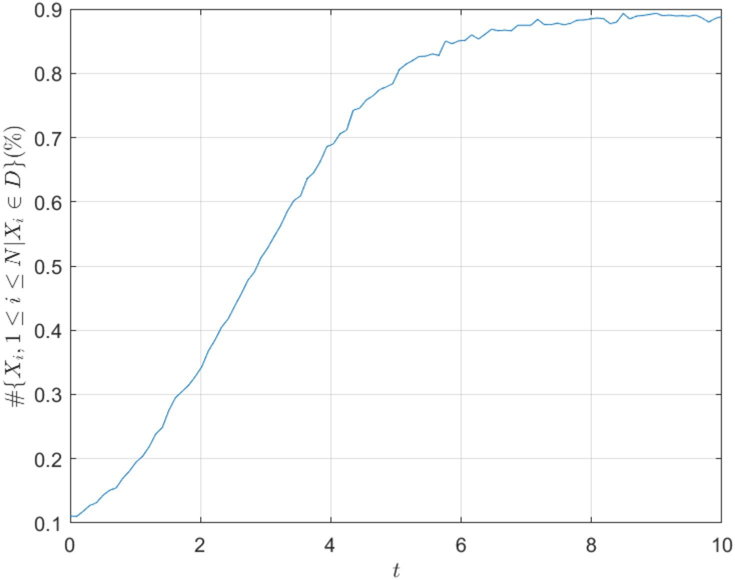

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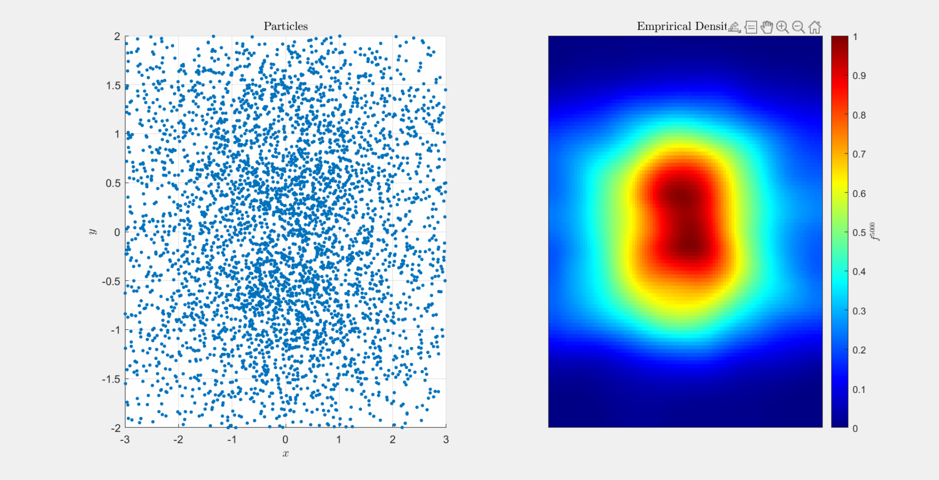

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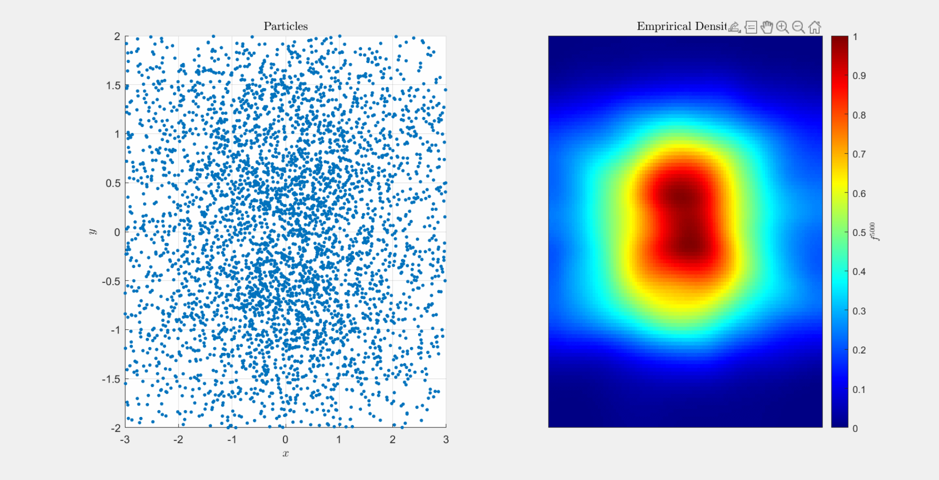

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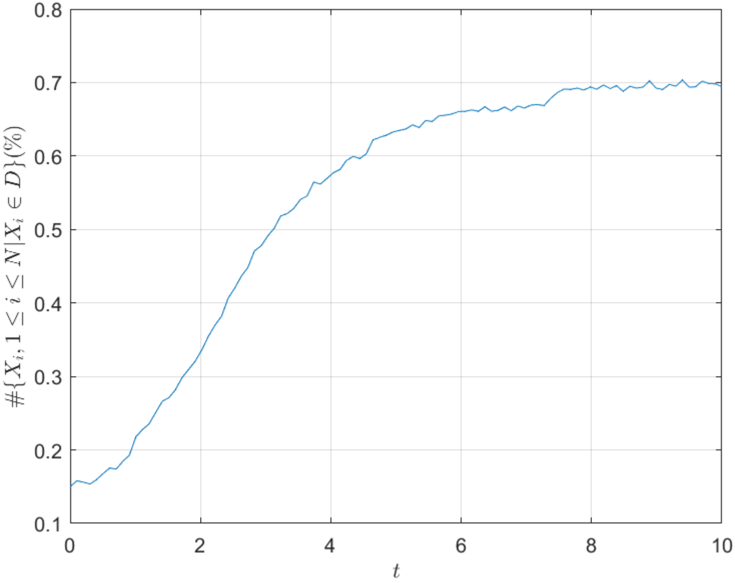
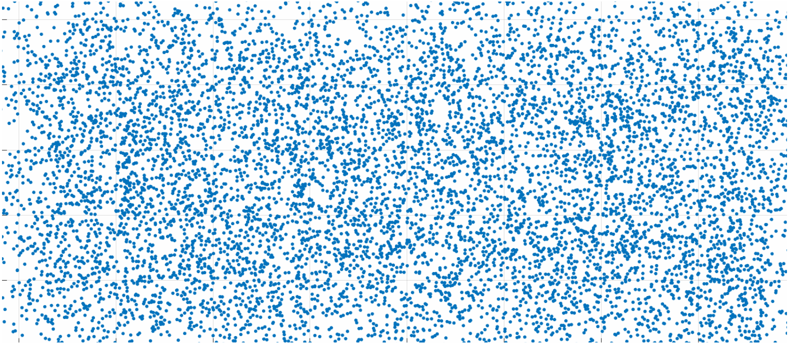


- We have investigated a simple system of SDEs to steer particles into a uniform distribution on a convex domain.
- For non-convex domains, we need the starting distribution to have certain properties.
- In the future, it would be interesting to look at systems of interacting particles, meaning that the drift term is of the form

$$\mu(X_1, \ldots, X_N, t).$$



Thanks for listening!

Questions?